KINETIC ENERGY, WORK, PRINCIPLE OF WORK AND ENERGY

Today’s objectives: Students will be able to

1. Define the various ways a force and couple do work.
2. Apply the principle of work and energy to a rigid body.

In-class activities:

- Reading Quiz
- Applications
- Kinetic Energy
- Work of a Force or Couple
- Principle of Work and Energy
- Concept Quiz
- Group Problem Solving
- Attention Quiz
1. Kinetic energy due to rotation (only) of the body is defined as
   (a) \( \frac{1}{2} m (v_G)^2 \)
   (b) \( \frac{1}{2} m (v_G)^2 + \frac{1}{2} I_G \omega^2 \)
   (c) \( \frac{1}{2} I_G \omega^2 \)
   (d) \( I_G \omega^2 \)
   ANS: __

2. When calculating work done by forces, the work of an internal force does not have to be considered because
   _______
   (a) internal forces do not exist
   (b) the forces act in equal but opposite collinear pairs
   (c) the body is at rest initially
   (d) the body can deform
   ANS: __
• The work of the torque (or moment) developed by the driving gears on the two motors on the concrete mixer is transformed into the rotational kinetic energy of the mixing drum.

• If the motor gear characteristics are known, how would you find the rotational velocity of the mixing drum?
The work done by the soil compactor’s engine is transformed into (1) the **translational** kinetic energy of the frame and (2) the translational and **rotational** kinetic energy of the roller and wheels (excluding the internal kinetic energy developed by the moving parts of the engine and drive train).

Are the kinetic energies of the frame and the roller related to each other? If so, how?
KINETIC ENERGY (Section 18.1)

• The kinetic energy of a rigid body can be expressed as the sum of its translational and rotational kinetic energies. In equation form, a body in general plane motion has kinetic energy given by:

\[ T = \frac{1}{2} m(\mathbf{v}_G)^2 + \frac{1}{2} \mathbf{I}_G \mathbf{\omega}^2 \]

• Several simplifications can occur.

1. (Pure) Translation: When a rigid body is subjected to only curvilinear or rectilinear translation, the rotational kinetic energy is zero \((\mathbf{\omega} = 0)\). Therefore,

\[ T = \frac{1}{2} m(\mathbf{v}_G)^2 \]
2. Pure Rotation: When a rigid body is rotating about a fixed axis passing through point $O$, the body has both translational and rotational kinetic energy. Thus,

$$T = \frac{1}{2}mv_G^2 + \frac{1}{2}I_G\omega^2$$

Since $v_G = r_G\omega$ where $r_G = OG$, we can express the kinetic energy of the body as:

$$T = \frac{1}{2}I_O\omega^2$$

which is the parallel axis theorem.

- If the rotation occurs about the mass center, $G$, then what is the value of $v_G$? In this case, the velocity of the mass center is equal to zero. So the kinetic energy equation reduces to: $T = \frac{1}{2}I_G\omega^2$. 
THE WORK OF A FORCE (Sec. 18.2)

• Recall that the work done by a force can be written as:
  \[ U_F = \int \mathbf{F} \cdot d\mathbf{r} = \int (F \cos \theta) ds \]

• When the force is constant, this equation reduces to
  \[ U_{Fc} = (F_c \cos \theta) s \] where \( F_c \cos \theta \) represents the component of the force acting in the direction of the displacement, \( s \).

• Work of a weight (\( W = mg \)): As before, the work can be expressed as \( U_W = -W \Delta y \) (+ ↑). Remember, if the force and movement are in the same direction, the work is positive.

• Work of a spring force: For a linear spring, the work is:
  \[ U_s = \int_{s_1}^{s_2} -kx \, dx = -\frac{1}{2} k[(s_2)^2 - (s_1)^2] \]
There are some external forces that do no work.

1. For instance, reactions at *fixed* supports do no work because the displacement at their point of application is *zero*.

2. Normal forces and friction forces acting on bodies as they roll *without slipping* over a rough surface also do *no* work since there is no instantaneous displacement of the point in contact with ground (it is an instant center, *IC*).

3. *Internal* forces do *no* work because they always act in *equal and opposite pairs*. Thus, the sum of their work is zero.
• When a body subjected to a couple experiences general plane motion, the two couple forces do work only when the body undergoes rotation.

• If the body rotates through an angular displacement $d\theta$, the work of the couple moment, $M$, is:

$$U_M = \int_{\theta_1}^{\theta_2} M \cdot d\theta$$

• If the couple moment, $M$, is constant, then

$$U_M = M(\theta_2 - \theta_1)$$

• Here, the work is positive, provided $M$ and $(\theta_2 - \theta_1)$ are in the same direction.
PRINCIPLE OF WORK AND ENERGY (Sec. 18.4)

• Recall the statement of the principle of work and energy used earlier:

\[ T_1 + \Sigma U_{1-2} = T_2 \]

• In the case of general plane motion, this equation states that the sum of the initial kinetic energy \((T_1, \text{ both translational and rotational})\) and the work done \((\Sigma U_{1-2})\) by all external forces and couple moments equals the body’s final kinetic energy \((T_2, \text{ translational and rotational})\).

• This equation is a scalar equation. It can be applied to a system of rigid bodies by summing contributions from all bodies.
EXAMPLE

• Given: The disk weighs 40 lb and has a radius of gyration \((k_G)\) of 0.6 ft. A 15 ft \(\cdot\) lb moment is applied and the spring has a spring constant of 10 lb/ft.

• Find: The angular velocity of the wheel when point \(G\) moves 0.5 ft. The wheel starts from rest and rolls without slipping. The spring is initially un-stretched.

• Plan: Use the principle of work and energy to solve the problem since distance is the primary parameter. Draw a free body diagram of the disk and calculate the work of the external forces.
EXAMPLE (Solution)

• Free body diagram of the disk ⇒
• Since the disk rolls without slipping on a horizontal surface, only the spring force and couple moment $M$ do work. Why don’t forces $F_B$ and $N_B$ do any work?
• Since the spring is attached to the top of the wheel, it will stretch twice the amount of displacement of $G$ (or 1 ft).
EXAMPLE (continued)

- **Work:**
  \[ U_{1-2} = -\frac{1}{2}k[s_2^2 - s_1^2] + M(\theta_2 - \theta_1) \]
  \[ U_{1-2} = -\frac{1}{2}(10)(1^2 - 0) + 15(0.5/0.8) \]
  \[ = 4.375 \text{ft} \cdot \text{lb} \]

- **Kinematic relation:** \( v_G = r\omega = \)

- **Kinetic energy:**
  \[ T_1 = 0 \]
  \[ T_2 = \frac{1}{2}m(v_G)^2 + \frac{1}{2}I_G\omega^2 \]
  \[ = \frac{1}{2}\left(\frac{40}{32.2}\right)(0.8\omega)^2 + \frac{1}{2}\left(\frac{40}{32.2}\right)(0.6)^2\omega^2 \]
  \[ = \]

- **Work and energy:** from \( T_1 + U_{1-2} = T_2 \), one gets
  \[ 0 + 4.375 = 0.621\omega^2 \], i.e., \( \omega = \)
CONCEPT QUIZ

1. If a rigid body rotates about its center of gravity, its translational kinetic energy is ______ at all times.
   (a) constant
   (b) zero
   (c) equal to its rotational kinetic energy
   (d) Cannot be determined
   ANS: ___

2. A rigid bar of mass m and length L is released from rest in the horizontal position. What is the rod’s angular velocity when it has rotated through 90°?

   \[ \theta = 90° \]

   (a) \( \sqrt{\frac{g}{3L}} \)
   (b) \( \sqrt{\frac{3g}{L}} \)
   (c) \( \sqrt{\frac{12g}{L}} \)
   (d) \( \sqrt{\frac{g}{L}} \)
   ANS: ___
GROUP PROBLEM SOLVING

• Given: The 50 kg pendulum of the Charpy impact machine is released from rest when $\theta = 0$. The radius of gyration $k_A = 1.75$ meter.

• Find: The angular velocity of the pendulum when $\theta = 90^\circ$.

• Plan: Since the problem involves distance, the principle of work and energy is an efficient solution method. The only force involved doing work is the weight, so only its work need be determined.
• Calculate the vertical distance the mass center moves. Because

\[ \Delta y = 1.25 \sin \theta \]

We can determine the work due to the weight \( (W = mg) \) from \( \theta = 0 \) to \( \pi/2 \).

\[
U_W = -W \Delta y
= 50(9.81)(0.0 - 1.25)
= \]

• The mass moment of inertia about A is:

\[ I_A = m(k_A)^2 = 50(1.75)^2 = \]
GROUP PROBLEM SOLVING (continued)

• Kinetic energy:

\[ T_1 = 0 \]
\[ T_2 = \frac{1}{2} I_A \omega^2 \]

• Now apply the principle of work and energy equation:

\[ T_1 + U_{1-2} = T_2 \]
\[ 0 + 613.1 = 76.55 \omega^2 \]
\[ \omega = \]
ATTENTION QUIZ

1. A disk and a sphere, each of mass \( m \) and radius \( r \), are released from rest. After 2 full turns, which body has a larger angular velocity? Assume roll without slip.

(a) Sphere  
(b) Disk  
(c) Equal.  
(d) Cannot

ANS: __

2. A slender bar of mass \( m \) and length \( L \) is released from rest in a horizontal position. The work done by its weight when it has rotated through \( 90^\circ \) is?

(a) \( mg(\pi/2) \)  
(b) \( mgL \)  
(c) \( mg(L/2) \)  
(d) \( -mg(L/2) \)

ANS: __
Today’s objectives: Students will be able to

1. Determine the potential energy of conservative forces.
2. Apply the principle of conservation of energy.

In-class activities:

• Reading Quiz
• Applications
• Potential Energy
• Conservation of Energy
• Concept Quiz
• Group Problem Solving
• Attention Quiz
1. Elastic potential energy is defined as _____.
   (a) $+\frac{1}{2}k(s)^2$
   (b) $-\frac{1}{2}k(s)^2$
   (c) $+\frac{1}{2}k(v)^2$
   (d) None of the above
   ANS: __

2. The kinetic energy of a rigid body consists of the kinetic energy due to _____.
   (a) translational motion and rotational motion
   (b) only rotational motion
   (c) only translational motion
   (d) the deformation of the body
   ANS: __
APPLICATIONS

- The *torsion* springs located at the top of the garage door wind up as the door is lowered.

- When the door is raised, the potential energy stored in the *spring* is transferred into the gravitational potential energy of the door’s *weight*, thereby making it easy to open.

- Are parameters such as the torsional spring stiffness and initial rotation angle of the spring important when you install a new door?
APPLICATIONS (continued)

• Two torsional springs are used to assist in opening and closing the hood of the truck.

• Assuming the springs are uncoiled when the hood is opened, can we determine the stiffness of each spring so that the hood can easily be lifted, i.e., practically no external force applied to it, when a person is opening it?

• Are the gravitational potential energy of the hood and the torsional spring stiffness related to each other? If so, how?
CONSERVATION OF ENERGY (Section 18.5)

• The conservation of energy theorem is a ‘simpler’ energy method (recall that the principle of work and energy is also an energy method) for solving problems.

• Once again, the problem parameter of distance is a key indicator for when conservation of energy is a good method to solve a problem.

• If it is appropriate for the problem, conservation of energy is easier to use than the principle of work and energy.

• This is because the calculation of the work of a conservative force is simpler. But, what makes a force conservative?
CONSERVATIVE FORCES

• A force $F$ is **conservative** if the work done by the force is **independent** of the path.
• In this case, the work depends only on the initial and final positions of the object with the path between the positions of **no** consequence.
• Typical conservative forces encountered in dynamics are **gravitational** forces (i.e., weight) and **elastic** forces (i.e., springs).
• What is a common force that is not conservative?
  ANS: *friction and drag*
CONSERVATION OF ENERGY

• When a rigid body is acted upon by a system of conservative forces, the work done by these forces is conserved. Thus, the sum of kinetic energy and potential energy remains constant. This principle is called conservation of energy and is expressed as:

\[ T_1 + V_1 = T_2 + V_2 = \text{Constant} \]

• In other words, as a rigid body moves from one position to another when acted upon by only conservative forces, kinetic energy is converted to potential energy and vice versa.
The gravitational potential energy of an object is a function of the height of the body’s center of gravity above or below a datum.

The gravitational potential energy of a body is found by the equation

\[ V_g = W y_G \]

Gravitational potential energy is positive when \( y_G \) is positive, since the weight has the ability to do positive work (why is it positive?) when the body is moved back to the datum.
• Spring forces are also conservative forces.

\[ V_e = + \frac{1}{2} ks^2 \]

• The potential energy of a spring force \( F = ks \) is found by the equation

\[ V_e = \frac{1}{2} ks^2 \]

• Notice that the elastic potential energy is always \textit{positive}. 
PROCEDURE FOR ANALYSIS

• Problems involving velocity, displacement and conservative force systems can be solved using the conservation of energy equation.

1 Potential energy: Draw two diagrams: one with the body located at its initial position and another at the final position. Compute the potential energy at each position using $V = V_g + V_e$, where $V_g = W y_G$ and $V_e = \frac{1}{2} k s^2$.

2 Kinetic energy: Compute the kinetic energy of the rigid body at each location. Kinetic energy has two components: translational kinetic energy, $\frac{1}{2} m v_G^2$, and rotational kinetic energy, $\frac{1}{2} I_G \omega^2$.

3 Apply the conservation of energy equation.
EXAMPLE I

• Given: The rod $AB$ has a mass of 10 kg. Piston $B$ is attached to a spring of constant $k = 800 \text{ N/m}$. The spring is un-stretched when $\theta = 0^\circ$. Neglect the mass of the pistons.

• Find: The angular velocity of rod $AB$ at $\theta = 0^\circ$ if the rod is released from rest when $\theta = 30^\circ$.

• Plan: Use the energy conservation equation since all forces are conservative and distance is a parameter (represented here by $\theta$). The potential energy and kinetic energy of the rod at states 1 and 2 will have to be determined.
EXAMPLE 1 (Solution): Potential Energy

- Initial Position

- Final Position

Let’s put the datum in line with the rod when $\theta = 0^\circ$. Then, the gravitational potential energy and the elastic potential energy will be zero at position 2: $V_2 =$

Gravitational potential energy at 1:

$$V_{g1} = -(10)(9.81)(\frac{1}{2}0.4 \sin 30^\circ)$$

Elastic potential energy at 1:

$$V_{e1} = \frac{1}{2}(800)(0.4 \sin 30^\circ)^2$$

So $V_1 = V_{g1} + V_{e1} =$
Initial Position

- The rod is released from rest from position 1: \( T_1 = 0 \).
- At position 2, the angular velocity is \( \omega_2 \) and the velocity at the center of mass is \( v_{G2} \):

\[
T_2 = \frac{1}{2} (10)(v_{G2})^2 + \frac{1}{2} \left( \frac{1}{12} \right) (10)(0.4^2)(\omega_2)^2
\]
EXAMPLE I (continued)

- At position 2, point \( A \) is the instantaneous center of rotation so that
  \[ v_{G2} = r_{G/IC} \omega = 0.2\omega \]
  and
  \[ T_2 = 0.2(\omega_2)^2 + 0.067(\omega_2)^2 \]

- Now apply the conservation of energy equation and solve for the unknown angular velocity, \( \omega_2 \).
  \[ T_1 + V_1 = T_2 + V_2 \]
  \[ 0 + 6.19 = 0.267(\omega_2)^2 + 0 \]
  \[ \omega_2 = \]
EXAMPLE II

• Given: The 30 kg rod is released from rest when $\theta = 0^\circ$. The spring is unstretched when $\theta = 0^\circ$.

• Find: The angular velocity of the rod when $\theta = 30^\circ$.

• Plan: Since distance is a parameter and all forces doing work are conservative, use conservation of energy. Determine the potential energy and kinetic energy of the system at both positions and apply the conservation of energy equation.
EXAMPLE II (Solution): Potential Energy

• Let’s put the datum in line with the rod when \( \theta = 0^\circ \). Then, the gravitational potential energy when \( \theta = 30^\circ \) is

\[
V_{g2} = -30(9.81)(1.5 \sin 30^\circ) \\
= -110.4 \text{N} \cdot \text{m}
\]

• The elastic potential energy at \( \theta = 0^\circ \) is zero since the spring is un-stretched. The un-stretched length of the spring is 0.5 meter. Elastic potential energy at \( \theta = 30^\circ \) is

\[
V_{e2} = \frac{1}{2}80(x - 0.5)^2 = 11.09 \text{N} \cdot \text{m}
\]
EXAMPLE II (continued): Kinetic Energy

• The rod is released from rest at \( \theta = 0^\circ \), so \( v_{G1} = 0 \) and \( \omega_1 = 0 \). Thus, the kinetic energy at position 1 is \( T_1 = 0 \).

• At \( \theta = 30^\circ \), the angular velocity is \( \omega_2 \) and the velocity at the center of mass is \( v_{G2} \).

• Since \( v_{G2} = \) ,

\[
T_2 = \frac{1}{2} m(v_{G2})^2 + \frac{1}{2} I_G(\omega_2)^2 \\
= \frac{1}{2} (30)(v_{G2})^2 + \frac{1}{2} (1/12)30(1.5)^2(\omega_2)^2 \\
= \frac{1}{2} (30)(0.75\omega_2)^2 + \frac{1}{2} (1/12)30(1.5)^2(\omega_2)^2 \\
= \]

\[11.25(\omega_2)^2\]
• Now all terms in the conservation of energy equation have been formulated. Writing the general equation and then substituting into it yields:

\[ T_1 + V_1 = T_2 + V_2 \]
\[ 0 + 0 = 11.25(\omega_2)^2 + (-110.4 + ) \]

• Solving for \( \omega_2 = \)
1. At the instant shown, the spring is undeformed. Determine the change in potential energy if the 20 kg disk \( k_G = 0.5 \text{ meter} \) rolls 2 revolutions without slipping.

(a) \( + \frac{1}{2} (200)(1.2\pi)^2 + (20)9.81(1.2\pi \sin 30^\circ) \)

(b) \( - \frac{1}{2} (200)(1.2\pi)^2 - (20)9.81(1.2\pi \sin 30^\circ) \)

(c) \( + \frac{1}{2} (200)(1.2\pi)^2 - (20)9.81(1.2\pi \sin 30^\circ) \)

(d) \( + \frac{1}{2} (200)(1.2\pi)^2 \)

ANS: ___

2. Determine the kinetic energy of the disk at this instant.

(a) \( \frac{1}{2}(20)(3)^2 \)

(b) \( \frac{1}{2}(20)(0.5^2)(10)^2 \)

(c) Answer (a) + Answer (b)

(d) None of the above

ANS: ___
GROUP PROBLEM SOLVING

• Given: The 30 kg pendulum has its mass center at $G$ and a radius of gyration about point $G$ of $k_G = 0.3$ meter. It is released from rest when $\theta = 0^\circ$. The spring is un-stretched when $\theta = 0^\circ$.

• Find: The angular velocity of the pendulum when $\theta = 90^\circ$.

• Plan: Conservative forces and distance ($\theta$) leads to the use of conservation of energy. First, determine the potential energy and kinetic energy for both positions. Then apply the conservation of energy equation.
Let’s put the datum when $\theta = 0^\circ$. Then, the gravitational potential energy and the elastic potential energy will be zero. So,

$$V_{g1} = V_{e1} = 0$$

Note that the un-stretched length of the spring is $0.15$ meter.

Gravitational potential energy at $\theta = 90^\circ$:

$$V_{g2} = -30(9.81)(0.35) =$$

Elastic potential energy at $\theta = 90^\circ$ is:

$$V_{e2} = \frac{1}{2}300(\sqrt{0.6^2 + 0.45^2} - 0.15)^2 =$$
• When $\theta = 0^\circ$, the pendulum is released from rest. Thus, $T_1 = 0$.

• When $\theta = 90^\circ$, the pendulum has a rotational motion about point $O$. Thus, $T_2 = \frac{1}{2} I_O (\omega_2)^2$

where

\[
I_O = I_G + m(d_{OG})^2 = (30)0.3^2 + 30(0.35)^2 = 6.375 \text{ kg} \cdot \text{m}^2
\]

\[
T_2 = \frac{1}{2} 6.375 (\omega_2)^2
\]

• Now, substitute into the conservation of energy equation.

\[
T_1 + V_1 = T_2 + V_2
\]

\[
0 + 0 = \frac{1}{2} 6.375 (\omega_2)^2 + (-103 + 54.0)
\]

• Solving for $\omega$ yields $\omega = \ldots$. 
ATTENTION QUIZ

1. Blocks A and B are released from rest and the disk turns 2 revolutions. The $V_2$ of the system includes a term for ....?

(a) only the 40 kg block
(b) only the 80 kg block
(c) the disk and both blocks
(d) only the two blocks

ANS: (d)

2. A slender bar is released from rest while in the horizontal position. The kinetic energy ($T_2$) of the bar when it has rotated through $90^\circ$ is?

(a) $\frac{1}{2}m(v_{G2})^2$
(b) $\frac{1}{2}I_G(\omega_2)^2$
(c) $\frac{1}{2}k(s_1)^2 - W(L/2)$
(d) $\frac{1}{2}mv_{G2}^2 + \frac{1}{2}I_G\omega_2^2$

ANS: (d)