

# Continuity, Navier-Stokes, and Stress Tensor Equations: Summary

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The tables below for continuity, Navier-Stokes, and stress equations are from “Transport Phenomena” written by Bird et al. [1960].

## References

R. Byron Bird, Warren E. Stewart, and Edwin N. Lightfoot. *Transport Phenomena*. John Wiley & Sons, Inc., New York, 1960.

TABLE 3.4-1  
THE EQUATION OF CONTINUITY IN SEVERAL  
COORDINATE SYSTEMS

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*Rectangular coordinates (x, y, z):*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) = 0 \quad (A)$$

*Cylindrical coordinates (r,  $\theta$ , z):*

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \quad (B)$$

*Spherical coordinates (r,  $\theta$ ,  $\phi$ ):*

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r}(\rho r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta}(\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi}(\rho v_\phi) = 0 \quad (C)$$

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TABLE 3.4-2  
THE EQUATION OF MOTION IN RECTANGULAR COORDINATES ( $x, y, z$ )

In terms of  $\tau$ :

$$\begin{aligned} x\text{-component} \quad \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= - \frac{\partial p}{\partial x} \\ &- \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) + \rho g_x \quad (A) \end{aligned}$$

$$\begin{aligned} y\text{-component} \quad \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= - \frac{\partial p}{\partial y} \\ &- \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \right) + \rho g_y \quad (B) \end{aligned}$$

$$\begin{aligned} z\text{-component} \quad \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= - \frac{\partial p}{\partial z} \\ &- \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \rho g_z \quad (C) \end{aligned}$$

In terms of velocity gradients for a Newtonian fluid with constant  $\rho$  and  $\mu$ :

$$\begin{aligned} x\text{-component} \quad \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= - \frac{\partial p}{\partial x} \\ &+ \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) + \rho g_x \quad (D) \end{aligned}$$

$$\begin{aligned} y\text{-component} \quad \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= - \frac{\partial p}{\partial y} \\ &+ \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) + \rho g_y \quad (E) \end{aligned}$$

$$\begin{aligned} z\text{-component} \quad \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= - \frac{\partial p}{\partial z} \\ &+ \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z \quad (F) \end{aligned}$$

TABLE 3.4-3  
THE EQUATION OF MOTION IN CYLINDRICAL COORDINATES  $(r, \theta, z)$

In terms of  $\tau$ :

$$\begin{aligned} r\text{-component}^a \quad \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = - \frac{\partial p}{\partial r} \\ - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} + \frac{\partial \tau_{rz}}{\partial z} \right) + \rho g_r \quad (A) \end{aligned}$$

$$\begin{aligned} \theta\text{-component}^b \quad \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = - \frac{1}{r} \frac{\partial p}{\partial \theta} \\ - \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} \right) + \rho g_\theta \quad (B) \end{aligned}$$

$$\begin{aligned} z\text{-component} \quad \rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} \\ - \left( \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rz}) + \frac{1}{r} \frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} \right) + \rho g_z \quad (C) \end{aligned}$$

In terms of velocity gradients for a Newtonian fluid with constant  $\rho$  and  $\mu$ :

$$\begin{aligned} r\text{-component}^a \quad \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = - \frac{\partial p}{\partial r} \\ + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right] + \rho g_r \quad (D) \end{aligned}$$

$$\begin{aligned} \theta\text{-component}^b \quad \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = - \frac{1}{r} \frac{\partial p}{\partial \theta} \\ + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] + \rho g_\theta \quad (E) \end{aligned}$$

$$\begin{aligned} z\text{-component} \quad \rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = - \frac{\partial p}{\partial z} \\ + \mu \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z \quad (F) \end{aligned}$$

TABLE 3.4-4  
THE EQUATION OF MOTION IN SPHERICAL COORDINATES  $(r, \theta, \phi)$

In terms of  $\tau$ :

$$\begin{aligned}
 \text{r-component} \quad & \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) \\
 & = -\frac{\partial p}{\partial r} - \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{r\theta} \sin \theta) \right. \\
 & \quad \left. + \frac{1}{r \sin \theta} \frac{\partial \tau_{r\phi}}{\partial \phi} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right) + \rho g_r \quad (A)
 \end{aligned}$$

$$\begin{aligned}
 \text{\theta-component} \quad & \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) \\
 & = -\frac{1}{r} \frac{\partial p}{\partial \theta} - \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial \tau_{\theta\phi}}{\partial \phi} \right. \\
 & \quad \left. + \frac{\tau_{r\theta}}{r} - \frac{\cot \theta}{r} \tau_{\phi\phi} \right) + \rho g_\theta \quad (B)
 \end{aligned}$$

$$\begin{aligned}
 \text{\phi-component} \quad & \rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi \cot \theta}{r} \right) \\
 & = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} - \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\phi}) + \frac{1}{r} \frac{\partial \tau_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \tau_{\phi\phi}}{\partial \phi} \right. \\
 & \quad \left. + \frac{\tau_{r\phi}}{r} + \frac{2 \cot \theta}{r} \tau_{\theta\phi} \right) + \rho g_\phi \quad (C)
 \end{aligned}$$

TABLE 3.4-4 (contd.)

In terms of velocity gradients for a Newtonian fluid with constant  $\rho$  and  $\mu$ :

$$\begin{aligned}
 \text{r-component} \quad & \rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) \\
 & = -\frac{\partial p}{\partial r} + \mu \left( \frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right) \\
 & \quad + \rho g_r \quad (D)
 \end{aligned}$$

$$\begin{aligned}
 \text{\theta-component} \quad & \rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot \theta}{r} \right) \\
 & = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} \right. \\
 & \quad \left. + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\phi}{\partial \phi} \right) + \rho g_\theta \quad (E)
 \end{aligned}$$

$$\begin{aligned}
 \text{\phi-component} \quad & \rho \left( \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r}{r} + \frac{v_\theta v_\phi \cot \theta}{r} \right) \\
 & = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \mu \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} \right. \\
 & \quad \left. + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial v_\theta}{\partial \phi} \right) + \rho g_\phi \quad (F)
 \end{aligned}$$

TABLE 3.4-5  
COMPONENTS OF THE STRESS TENSOR FOR NEWTONIAN FLUIDS  
IN RECTANGULAR COORDINATES  $(x, y, z)$

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$\tau_{xx} = -\mu \left[ 2 \frac{\partial v_x}{\partial x} - \frac{2}{3}(\nabla \cdot v) \right]$	(A)
$\tau_{yy} = -\mu \left[ 2 \frac{\partial v_y}{\partial y} - \frac{2}{3}(\nabla \cdot v) \right]$	(B)
$\tau_{zz} = -\mu \left[ 2 \frac{\partial v_z}{\partial z} - \frac{2}{3}(\nabla \cdot v) \right]$	(C)
$\tau_{xy} = \tau_{yx} = -\mu \left[ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right]$	(D)
$\tau_{yz} = \tau_{zy} = -\mu \left[ \frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right]$	(E)
$\tau_{zx} = \tau_{xz} = -\mu \left[ \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right]$	(F)
$(\nabla \cdot v) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$	(G)

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TABLE 3.4-6  
COMPONENTS OF THE STRESS TENSOR FOR NEWTONIAN FLUIDS  
IN CYLINDRICAL COORDINATES  $(r, \theta, z)$

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$\tau_{rr} = -\mu \left[ 2 \frac{\partial v_r}{\partial r} - \frac{2}{3}(\nabla \cdot v) \right]$	(A)
$\tau_{\theta\theta} = -\mu \left[ 2 \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) - \frac{2}{3}(\nabla \cdot v) \right]$	(B)
$\tau_{zz} = -\mu \left[ 2 \frac{\partial v_z}{\partial z} - \frac{2}{3}(\nabla \cdot v) \right]$	(C)
$\tau_{r\theta} = \tau_{\theta r} = -\mu \left[ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]$	(D)
$\tau_{\theta z} = \tau_{z\theta} = -\mu \left[ \frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right]$	(E)
$\tau_{zr} = \tau_{rz} = -\mu \left[ \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right]$	(F)
$(\nabla \cdot v) = \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$	(G)

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TABLE 3.4-7  
 COMPONENTS OF THE STRESS TENSOR FOR NEWTONIAN FLUIDS  $S$   
 IN SPHERICAL COORDINATES  $(r, \theta, \phi)$

$$\tau_{rr} = -\mu \left[ 2 \frac{\partial v_r}{\partial r} - \frac{2}{3}(\nabla \cdot v) \right] \quad (A)$$

$$\tau_{\theta\theta} = -\mu \left[ 2 \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) - \frac{2}{3}(\nabla \cdot v) \right] \quad (B)$$

$$\tau_{\phi\phi} = -\mu \left[ 2 \left( \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right) - \frac{2}{3}(\nabla \cdot v) \right] \quad (C)$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu \left[ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (D)$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu \left[ \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] \quad (E)$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu \left[ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \right] \quad (F)$$

$$(\nabla \cdot v) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \quad (G)$$

TABLE 3.4-8  
THE FUNCTION  $-(\boldsymbol{\tau} : \nabla v) = \mu \Phi_v$  FOR NEWTONIAN FLUIDS\*

<i>Rectangular</i>	$\Phi_v = 2 \left[ \left( \frac{\partial v_x}{\partial x} \right)^2 + \left( \frac{\partial v_y}{\partial y} \right)^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 \right]$ $+ \left[ \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right]^2 + \left[ \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right]^2 + \left[ \frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right]^2$ $- \frac{2}{3} \left[ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]^2 \quad (A)$
<i>Cylindrical</i>	$\Phi_v = 2 \left[ \left( \frac{\partial v_r}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left( \frac{\partial v_z}{\partial z} \right)^2 \right]$ $+ \left[ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2 + \left[ \frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right]^2$ $+ \left[ \frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right]^2$ $- \frac{2}{3} \left[ \frac{1}{r} \frac{\partial}{\partial r} (r v_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \right]^2 \quad (B)$
<i>Spherical</i>	$\Phi_v = 2 \left[ \left( \frac{\partial v_r}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 \right]$ $+ \left( \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right)^2$ $+ \left[ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2$ $+ \left[ \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right]^2$ $+ \left[ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \right]^2$ $- \frac{2}{3} \left[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right]^2 \quad (C)$