CEE 618 spring 2011 HW #6 Prob. 1

1. $$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0 \qquad \text{— (1)}$$

Boundary conditions

$C(0, y) = 0$ — (2)

$C(x, y=0) = 0$ — (3)

$C(x, y=L) = 0$ — (4)

$C(L, y) = 20 \sin(2\pi y/L)$ — (5)

Solution using the method of separation of variables

· Set $C(x,y) = C_x(x) \times C_y(y)$

· Then, Eq. (1) is

$$C_y \frac{d^2 C_x}{dx^2} + C_x \frac{d^2 C_y}{dy^2} = 0 \qquad \div\ C_x \cdot C_y$$

$$\frac{1}{C_x}\frac{d^2 C_x}{dx^2} = -\frac{1}{C_y}\frac{d^2 C_y}{dy^2} \qquad \text{— (6)}$$

In Eq. (6), the left-hand-side is a sole function of x, and the right-hand-side of y. To validate the equality, they should be a constant, let's say k^2.

For C_x : $$\frac{d^2 C_x}{dx^2} - k^2 C_x = 0 \qquad \text{— (7)}$$

For C_y : $$\frac{d^2 C_y}{dy^2} + k^2 C_y = 0 \qquad \text{— (8)}$$

The general solutions are

$C_x = a_0 \cosh(kx) + b_0 \sinh(kx)$ — (9)

$C_y = c_0 \cos(ky) + d_0 \sin(ky)$ — (10)

There fore,

$$C = C_x(x)\, C_y(y)$$
$$= A_1 \cosh(kx)\cdot\cos(ky)$$
$$+ A_2 \cosh(kx)\cdot\sin(ky)$$
$$+ A_3 \sinh(kx)\cdot\cos(ky)$$
$$+ A_4 \sinh(kx)\cdot\sin(ky)$$

The four unknown constants, $A_1 - A_4$ can be determined using four boundary conditions.

BC 1 : Eq (2).

$$C(0, y) = 0 = A_1 \cos(ky) + A_2 \sin(ky)$$

Because y varies from 0 to L, to satisfy this condition, $A_1 = 0$ & $A_2 = 0$. Now, the general solution reduces to

$$C(x, y) = \sinh(kx)\,[A_3 \cos(ky) + A_4 \sin(ky)]$$

BC. 2 : Eq (3)

$$C(x, y=0) = 0 = \sinh(kx)\,[A_3\cdot 1 + A_4\cdot 0] = 0.$$

Therefore, $A_3 = 0$, and

$$C(x, y) = A_4 \sinh(kx) \sin(ky)$$

BC 3 : Eq (4)

$$C(x, y=L) = 0 = A_4 \sinh(kx) \sin(kL) = 0.$$

Here, A_4 cannot be zero, because if so, we get $C(x, y) = 0$ which is a trivial solution. To satisfy this condition,

$$kL = 0, \pi, 2\pi, \cdots = n\pi$$

where $n = 0, 1, 2, \cdots$. Here we define $k_n = n\pi/L$.

Then, the general solution can be

$$C(x,y) = \sum F_n \sinh(k_n x) \sin(k_n y)$$

where F_n is a coefficient that is a replacement of A_4.

BC 4: Eq (5)

$$\begin{aligned} C(L,y) &= \sum_{n=1}^{\infty} F_n \sinh(k_n x) \sin(k_n y) \\ &= F_1 \sinh(k_1 x)\cdot \sin(k_1 y) + F_2 \sinh(k_2 x)\cdot \sin(k_2 y) \\ &\quad + \cdots \\ &= 20 \sin\left(\frac{2\pi}{L} y\right) \\ &= 20 \sin(k_2 y) \end{aligned}$$

In general, F_n should be calculated using the orthogonality of sine and cosine functions, but in this case we can easily set

$$F_2 = 20 \quad \text{and} \quad F_1 = F_3 = F_4 = \cdots = 0.$$

Then the final solution is

$$C(x,y) = 20 \sinh(2\pi x/L)\cdot \sin(2\pi y/L)$$

Which indicates $C(x,y)$ monotonously increases along x-direction and is periodic in y-direction. Choice of trigonometric and hyperbolic functions in x- and y-direction is based on the sign of k^2. Of one changes $k^2 \to -k^2$ or $k \to ik$ where $i = \sqrt{-1}$, then C_x and C_y contains sin/cos and sinh/cosh functions.