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Effect of porous support fabric on osmosis through a Loeb–Sourirajan type asymmetric membrane

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Abstract

Commercially available asymmetric membranes of the Loeb–Sourirajan (L–S) type comprise a support fabric, bonded to the porous substructure. The influence of this fabric on osmotic permeation flux was examined, mostly with a Toray CA-3000 membrane from which, with care, it was possible to remove the support fabric. In osmosis experiments with 12% $MgCl_2$ solution on one side (either side) and 6% solution on the other, the permeation flux (J_1) was of the order of 0.01 and 0.06 m^3/m^2 d with and without fabric, respectively. These results could be generalized by considering the resistivity to solute diffusion in the non-skin part of the membrane. This resistivity term averaged 104 and 17 d/m for membranes with and without fabric, respectively, and in further tests without fabric, it was between 15 and 25 d/m over a wide range of $MgCl_2$ concentrations. Four other L–S membranes, all with support fabric, were tested in osmosis experiments. Their resistivity values were similar to or higher than those of the Toray membrane with fabric, but, with one of the four, the results were affected by switching the location of the high and low concentration solutions. It was concluded that existing commercially available L–S membranes are not appropriate for large-scale osmosis applications because their support fabric decreases permeation flux excessively.

Keywords: Asymmetric membranes; Concentration polarization; Loeb–Sourirajan membranes; Membrane preparation and structure; Osmosis

1. Introduction

The practical large-scale use of pressure-retarded osmosis (PRO) has been considered by previous investigators [1,2] who concluded that among the commercially available flat sheet membranes, the Loeb–Sourirajan (L–S) type holds the most promise for osmosis because it offers the highest permeation flux for a given osmotic driving force.

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These investigators also found that the porous substructure of such membranes is the principal hindrance to osmotic flow. However, their work was always confined to membranes without the woven or non-woven porous support fabric which is universally applied today to all commercially available L–S membranes by bonding to the substructure side.

Since the porous substructure is the principal resistance to osmotic permeation through a membrane without a support fabric, one must ask whether the addition of a fabric on the porous substructure side would act as an excessive additional resistance. The

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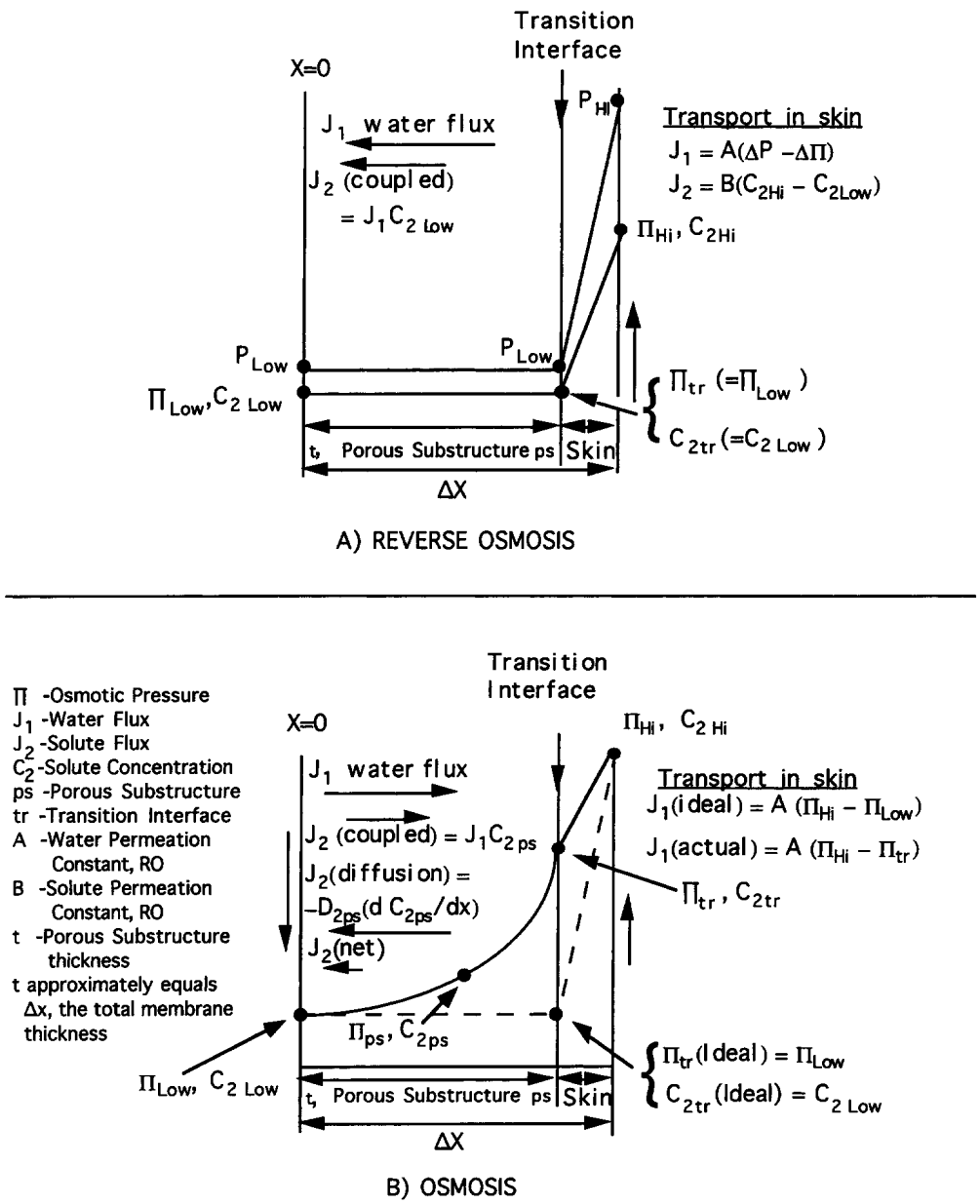


Fig. 1. Transport conditions in (A) reverse osmosis and (B) osmosis, both through an asymmetric membrane.

purpose of this paper is to provide information relevant to this question.

2. Comparison of transport in reverse osmosis and osmosis, both through an asymmetric membrane without a support fabric

Fig. 1A shows the pressure and solute gradients across an L–S membrane in *reverse osmosis* (RO). The water permeation flux, J_1, is taken from the high concentration solution, flows to the left from the skin into the porous substructure, and passes through it with no change in the solute concentration, C_2, and little change in pressure. Hence, the porous substructure thickness plays only a minor role in determining the magnitude of J_1.

In *osmosis*, the situation has been described by Lee et al. [2] as internal polarization. As shown in Fig. 1B[1], the water permeation flux, J_1, is taken from the low concentration solution, and flows to the right in the porous substructure, carrying the solute with it. At any given point in the porous substructure, where the solute concentration is C_{2ps}, the solute flux, J_2, tends to flow to the right, while coupled to J_1, at a rate $J_2 = J_1 C_{2ps}$. The solute flux is largely stopped by the skin at the transition interface, causing the solute concentration there, C_2, to rise. However, this rise causes back diffusion of the solute to the left at a rate $J_2 = -(D_{2ps})(dC_2/dx)$ where D_{2ps} is the diffusivity of the solute in the porous substructure.

The solute concentration, C_2, continues to rise until a steady state is reached at π_{tr} and C_{2tr}. Since π_{tr} is greater than π_{Low}, the skin transport equations in Fig. 1B show that J_1 (actual) is less than J_1 (ideal). J_1 (actual) can only approach closely to J_1 (ideal) if the solute back diffusion rate is high or the diffusion thickness, t, is low. This is equivalent to saying that the solute resistivity K $(= t/D_{2ps})$ in the substructure is low. Thus, in osmosis, the porous substructure clearly has a large effect in determining the water permeation flux, J_1.

The exact equation, developed by Lee et al. [2], is given in Appendix A for the case in which the low concentration solution is on the porous substructure side. If the not-always-precise assumption, $C_{2A}/C_{2B} = \pi_A/\pi_B$, is made and the experimentation is restricted to the condition of appreciable osmotic pressure on both sides of the membrane, the Lee equation can be reduced to Eq. (A3) of Appendix A:

$$K = (1/J)\ln(\pi_{Hi}/\pi_{Low})$$

It appears from further analysis that this equation would apply whether the low concentration solution is on the skin side or the porous substructure side of the membrane. It should also be noted that the osmotic driving force is essentially the logarithm of the ratio of high to low osmotic pressure. In RO, the osmotic driving force is the difference between high and low osmotic pressures.

The above discussion and the analysis in Appendix A are for the asymmetric membrane without a support fabric, in which case the solute resistivity term, K, refers to solute diffusion in the porous substructure. When the support fabric is included, there are, as shown in Fig. 2, three distinct regions for solute diffusion specifically: most of the substructure, most of the support fabric, and an overlap region combining the first two.

The methods of testing and data analysis used here are also appropriate for an overall solute resistivity

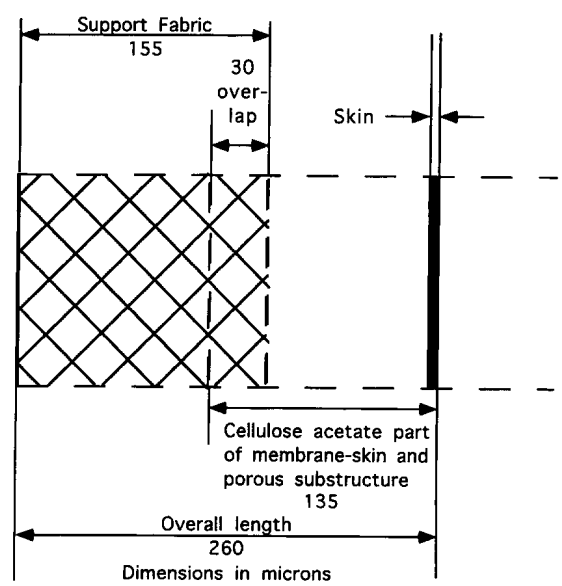


Fig. 2. Edgewise dimensions of the Toray CA-3000 membrane.

[1]Fig. 1B also gives nomenclature used throughout this paper with the exception of K_{fab}, to be defined in this section.

term to be used with the membranes having a support fabric. This resistivity term will be called K_{fab}.

3. Description of Toray CA-3000 membrane

The Toray CA 3000 membrane is an RO cellulose acetate asymmetric membrane supported by a woven fabric. It was chosen for extensive tests because the support fabric could, with care, be removed from the rest of the membrane. This enabled comparison of performance with and without the fabric. In an edge-wise view, the membrane and support material have the dimensions shown in Fig. 2.

The characteristic performance of the membrane in RO, according to Toray, is: fractional NaCl rejection, R, 0.98 (98%); water flux, J_1, 0.90 m^3/m^2 d. Data were obtained at a feed concentration of 1500 mg/l NaCl, and operating pressure of 30 kg/cm^2 (or 29.03 atm or 29.41 bar or 426.8 psi).

NaCl at 1500 mg/l has 1.2 atm osmotic pressure. Then, in RO:

$A = J_1/(\Delta P - \Delta\pi) = 0.9/(29.03 - 1.2) = 0.0328 m^3/m^2$ d atm.

$B = J_2/(C_{2Hi} - C_{2Low})$. B is the solute permeation constant, in m/d. If C_2 has the units kg of solute per m^3 then J_2 is kg of solute per m^2 per day. It can be shown [2] that:

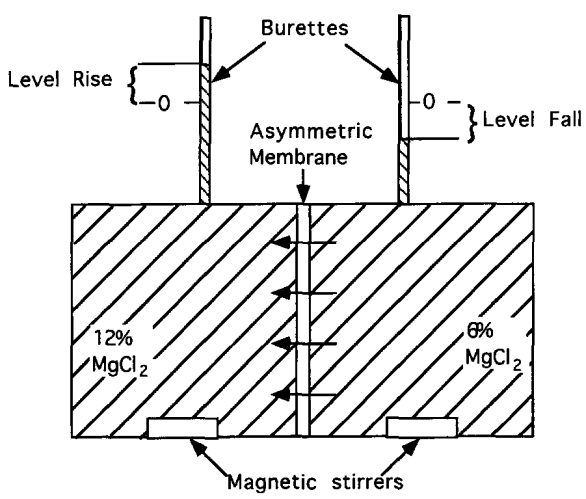
$$B = \frac{(1-R)(A)(\Delta P - \Delta\pi)}{R} = \frac{(1-R)J_1}{R} = \frac{(1-0.98)(0.90)}{0.98} = 0.0184\,\mathrm{m/d}$$

4. Test apparatus and procedure

The osmosis test apparatus is shown conceptually in Fig. 3. As a batch device, conditions were changing constantly during the course of a test. Specifically, the dilute solution was concentrating and the other was diluting. Hence, the osmotic driving force, $\ln(\pi_{Hi}/\pi_{Low})$, was continually diminishing, producing a corresponding decrease in the water permeation flux, J_1, in accordance with Eq. (A3) in Appendix A:

$$J_1 \approx \ln(\pi_{Hi}/\pi_{Low})$$

The tests were short enough so that the arithmetic average values were acceptable, i.e. $\ln(\pi_{Hi,ave}/\pi_{Low,ave})$. J_1 was determined of course from the change in burette levels. The tests reported herein were all made with appreciable magnesium chloride concentrations on both sides of the membranes. The con-



Conditions tested:
1) Low concentration solution on porous substructure side or on skin side.
2) Solution pairs 6-12% (as shown) or 6-18, 12-18, 12-24% $MgCl_2$.
3) Membrane with or without fabric.

Fig. 3. Osmosis test cell.

centrations and therefore osmotic pressures were readily known from density measurements of solutions both before and after the tests.

5. Purposes of tests

The primary purpose of the tests was to determine the effect of the support fabric on osmotic water permeation flux. The value of the solute resistivity term, K (or K_{fab}), was also determined for each test. Since K is a property dependent primarily on the nature of the solute and the porous medium through which it diffuses, K should be relatively constant for a given solute whatever the value of J_1, π_{Hi}, and π_{Low} in a given test, i.e. K should be an appropriate parameter for evaluating osmosis membranes. Hence, another purpose of these tests was to determine the constancy of K with varying values of J_1, π_{Hi}, and π_{Low}.

Another tentative conclusion reached in the development of Eq. (A3) was that it should be valid whether the low concentration solution is on the skin side or the opposite side of the membrane, with or without fabric. This conclusion could also be examined by the test procedures, i.e. by simply switching the solutions from one side to the other.

6. Test results with Toray cellulose acetate membrane, CA-3000

The following conclusions can be gleaned from the tests reported in Table 1:

First, the detrimental effect of the support fabric on the water permeation flux is very clear. In the tests with the 12/6 concentration pair the average permeation flux was ~1/6 as high with the fabric as without it (0.01 vs. 0.06 m^3/m^2 d). Approximately, this same value is seen in the 17/104 ratio for K/K_{fab}.

Second, in all the tests made without the support fabric, the term K was relatively constant at values between 15 and 25 d/m, regardless of appreciable variation in J_1, π_{Hi}, and π_{Low}, and their ratio. Thus, Eq. (A3) is supported and therefore the expectation

Table 1
Effect of support fabric on osmosis through the Toray CA-3000 membrane

Test dates (November 1995) [a]	Approximate initial $MgCl_2$ solutions on each side of the membrane		Water permeation flux J_1 (m^3/m^2 d)	Resistivity to $MgCl_2$ diffusion (d/m)		Solution in contact with skin side, Hi or Low concentration
	Concentration ratio C_{2Hi}/C_{2Low} (%/%)	Osmotic pressure π_{Hi}/π_{Low} (atm/atm)		With fabric, K_{fab}	Without fabric, K	
1	12/6	140/50	0.012	101		Hi
22			0.0097	112		Low
20			0.0097	98		Hi
				Ave. 104		
8	12/6	140/50	0.060		18	Hi
5			0.056		15	Low
16			0.051		18	Hi
2			0.071		15	Hi
12			0.064		18	Hi
					Ave. 17	
9	18/6	290/50	0.094		17	Hi
13	18/12	290/140	0.020		25	Hi
23			0.031		23	Hi
15	24/12	540/140	0.059		20	Hi

[a] Tests on 8, 9, 12, 13, 15 and 16 November were all made with the same membrane.

that the term K is dominant and general in determining the osmotic permeation flux.

Third, the tests on 22nd and 5th November were done with the low concentration solution on the skin side. The approximate equality of K and K_{fab} values with those of the other tests in their respective groups supported the belief that the side of the membrane in contact with the low concentration solution is not important (but see the following section).

7. Tests with other cellulose acetate membranes

Osmosis tests with 6 and 12% $MgCl_2$ solutions were made on several other commercially available cellulose acetate membranes. All of these were backed with woven or non-woven fabric. No effort was made to remove the fabric.

The results given in Table 2 show that the K_{fab} values are in the range of those with the fabric-backed Toray membrane or higher. As with the Toray membrane, the results were independent of the location of the 6 and 12% $MgCl_2$ solutions for the Fluid Systems', Trisep's, and Desalination Systems' membranes. However, the Hydranautics' membrane had roughly twice the K_{fab} value when the low concentration solution was against the skin as when the solutions were reversed. This anomalous behavior was not explored further.

Table 2
Solute resistivity (K_{fab}) values obtained with fabric-backed cellulose acetate membranes other than Toray CA-3000

Manufacturer	Designation	K_{fab}, (d/m)	Commentary
Fluid Systems	LP	82	Note 1
	CA/CTA	96	Note 2
Trisep	None	140	Note 1
		140	Note 2
Desalination Systems	CD-Yuma	380	Note 1
	CA/CTA	380	Note 2
Hydranautics	CAB-1	246, 204	Note 1
		93, 136	Note 2

Note 1: Low concentration solution against skin.
Note 2: Low concentration solution against fabric.
All tests done with 6 and 12% $MgCl_2$.

8. Discussion and conclusions

Clearly, the porous support fabric on L–S membranes decreases osmotic permeation excessively. However, without the fabric the membrane is too weak. A compromise would appear to be the use of a much more open fabric. However, the unanimous opinion of the manufacturers is that a more open fabric would cause 'bleeding' through it during fabrication, and would decrease the reliability of the completed membrane.

As touched upon by Lee et al. [2], an L–S capillary tube might be appropriate since it would not require further support. The internal diameter should be at least 100 μm to minimize axial pressure drop within the capillary tube. The wall thickness should be great enough to maintain capillary rigidity but otherwise, thin enough to achieve adequate permeation flux since K is proportional to porous substructure thickness. A module filled with capillary tubing would be satisfactory for osmosis with non-precipitating solutions. However, in one possible application, the concentration of Dead Sea brine, extensive precipitation of NaCl would occur as the brine is dewatered because it is saturated to start. Therefore, the module would have to be designed to pass a slurry of concentrated Dead Sea brine and NaCl without clogging, which is not an easy task.

Acknowledgements

We wish to thank the Toray, Fluid Systems, Trisep, and Hydranautics companies for the opportunity to test their membranes.

Appendix A

Key equations without support fabric

A.1 Low concentration solution on porous substructure side of membrane

Eq. (10) in [2] is:

$$\frac{J_1}{A\pi_{Hi}} = \frac{1 - (C_{2Low}/C_{2Hi})\exp(J_1K)}{1 + B/J_1[\exp(J_1K) - 1]} \quad \text{(A1)}$$

In this equation, A and B are known from the RO experiments; π_{Hi}, C_{2Low}, and C_{2Hi} are given conditions

in the osmosis experiment; and J_1 is measured during the experiment. Therefore, the equation can be used to determine K, defined as t/D_{2ps}.

By using the rough but simplifying assumption that for a given solute in a solution, $\pi_A/\pi_B = C_{2A}/C_{2B}$, Eq. (A1) can be simplified to:

$$K = (1/J_1)\left(\ln\frac{B + A\pi_{Hi} - J_1}{B + A\pi_{Low}}\right) \quad \text{(A2)}$$

In our tests, π_{Low} was always appreciably higher than zero. Also, $A\pi_{Low}$ and, therefore, $A\pi_{Hi}$ were much higher than B or J, so that we can write:

$$K = 1/J_1\left(\ln\frac{\pi_{Hi}}{\pi_{Low}}\right) \quad \text{(A3)}$$

A.2 Low concentration solution on skin side of membrane

By an analysis similar to that in Section A.2 above, we arrive at:

$$K = (1/J_1)\left(\ln\frac{B + A\pi_{Hi}}{B + J_1 + A\pi_{Low}}\right) \quad \text{(A4)}$$

Using the same simplifying assumptions that led to Eq. (A3), we arrive at the same equation. Therefore, the same value of K should be obtained whichever side of the membrane is in contact with the low concentration solution. Note that Eq. (A3) can be written to emphasize the inverse effect of substructure thickness t on J_1:

$$J_1 = \frac{1}{K}\left(\ln\frac{\pi_{Hi}}{\pi_{Low}}\right) = \frac{D_{2ps}}{t}\left(\ln\frac{\pi_{Hi}}{\pi_{Low}}\right)$$

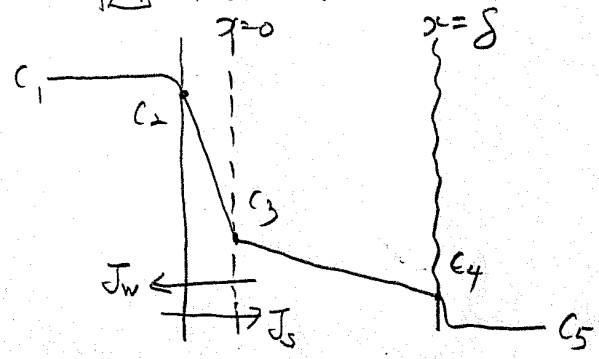
References

[1] S. Loeb, Pressure-Retarded Osmosis Revisited: The Prospects for Osmotic Power at the Dead Sea, Proceedings of Euromembrane '95, Vol. 2, University of Bath, September 1995, pp. II-3–II-11.

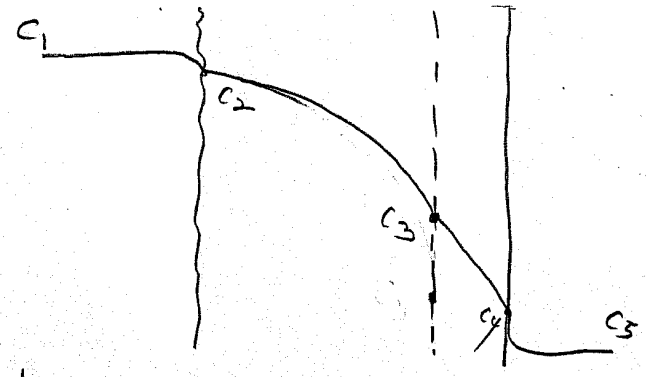
[2] K. Lee, R. Baker and H. Lonsdale, Membranes for power generation by pressure-retarded osmosis, J. Membrane Sci., 8 (1981) 141–171.

Internal Concentration Polarization of Forward Osmosis

$\boxed{A}$. Lee et al.



$\boxed{B}$. Loeb et al.



J_w = water flux
J_s = solute flux) in magnitude

A. Lee et al. : $J_w = ?$

- $J_w = A\,\pi_{eff} = A(\pi_2 - \pi_3)$ —① : water flux
- $J_s = B\,(C_2 - C_3)$ —② : solute flux
- In support layer

$$J_s = -D\frac{\varepsilon}{\tau}\frac{dC}{dx} - J_w C \quad \text{—③ : Convective diffusion eq.}$$

Solution.

$$D\frac{\varepsilon}{\tau}\frac{dC}{dx} = -J_w C - J_s$$

$$\frac{dC}{dx} = -\frac{\tau J_w}{D\varepsilon}\underbrace{\left(C + J_s/J_w\right)}_{=C'}$$

$$\Rightarrow \frac{dC'}{dx} = -\frac{\tau J_w}{D\varepsilon}C'$$

$$\ln C' = C_0' - \frac{J_w \tau}{D\varepsilon}x$$

$$C = -J_s/J_w + C_0''\,e^{-\frac{J_w \tau x}{D\varepsilon}}$$

B.C. $C(x=0) = C_3$

$$-J_s/J_w + C_0''\cdot 1 = C_3$$

$$\therefore C_0'' = C_3 + J_s/J_w$$

$$\Rightarrow C(x) = -J_s/J_w + (J_s/J_w + C_3)\,e^{-\frac{J_w \tau x}{D\varepsilon}}$$

B.C $C(x=\delta) = C_4$

$$C_4 = -J_s/J_w + (J_s/J_w + C_3)\,e^{-J_w K}$$

where $K = \frac{\tau\delta}{D\varepsilon}$

$$\boxed{C_4 = (J_s/J_w)(e^{-J_w K} - 1) + C_3 e^{-J_w K}} \quad \text{—④}$$

water flux J_w

$$J_w = A(\Pi_2 - \Pi_3)$$

$$= \quad \text{Assume} \quad \frac{\Pi_2 - \Pi_3}{\Pi_2 - \Pi_4} = \frac{C_2 - C_3}{C_2 - C_4}$$

$$= A \cdot \frac{C_2 - C_3}{C_2 - C_4}(\Pi_2 - \Pi_4)$$

$$= A \frac{\Pi_2(1 - \Pi_4/\Pi_2)}{(1 - C_2/C_4)}\left(1 - \frac{C_3}{C_2}\right)$$

$$= A\Pi_2\left(1 - \frac{C_3}{C_2}\right) \quad \text{—(5)}$$

From Eq. (4)

$$C_4 = \frac{J_s}{J_w}(e^{-J_w k} - 1) + C_3 e^{-J_w k}$$

$$J_w C_4 = B(C_2 - C_3)(e^{-J_w k} - 1) + J_w C_3 e^{-J_w k}$$

$$= BC_2(e^{-J_w k} - 1) - BC_3(e^{-J_w k} - 1) + J_w C_3 e^{-J_w k}$$

$$J_w C_4 - BC_2(e^{-J_w k} - 1) = C_3\left[J_w e^{-J_w k} - B(e^{-J_w k} - 1)\right]$$

$$C_3 = \frac{J_w C_4 - BC_2(e^{-J_w k} - 1)}{J_w e^{-J_w k} - B(e^{-J_w k} - 1)}$$

$$= \frac{J_w C_4 e^{J_w k} - BC_2(1 - e^{J_w k})}{J_w - B(1 - e^{J_w k})}$$

$$C_3 = \frac{J_w C_4 e^{J_w k} + BC_2(e^{J_w k} - 1)}{J_w + B(e^{J_w k} - 1)}$$

$$\frac{C_3}{C_2} = \frac{(C_4/C_2) J_w e^{J_w k} + B(e^{J_w k} - 1)}{J_w + B(e^{J_w k} - 1)} \quad \text{—(6)}$$

$$\cdot\ 1 - \frac{C_3}{C_2}$$

$$= \frac{J_w + B(e^{J_w k} - 1) - B(e^{J_w k} - 1) - \frac{C_4}{C_2} J_w e^{J_w k}}{J_w + B(e^{J_w k} - 1)}$$

$$= \frac{J_w - \frac{C_4}{C_2} J_w e^{J_w k}}{J_w + B(e^{J_w k} - 1)}$$

$$= \frac{J_w\left(1 - \frac{C_4}{C_2} e^{J_w k}\right)}{J_w + B(e^{J_w k} - 1)}$$

Then,

$$J_w = A\Pi_2\left(1 - \frac{C_3}{C_2}\right)$$

$$= \frac{A\Pi_2 J_w\left(1 - \frac{C_4}{C_2} e^{J_w k}\right)}{J_w + B(e^{J_w k} - 1)}$$

$$\Rightarrow J_w\left[J_w + B(e^{J_w k} - 1)\right]$$

$$= A\Pi_2 J_w\left(1 - \frac{\Pi_4}{\Pi_2} e^{J_w k}\right)$$

$$= A J_w(\Pi_2 - \Pi_4 e^{J_w k})$$

$$J_w + B\left[e^{J_w k} - 1\right]$$
$$= A\left(\Pi_2 - \Pi_4 e^{J_w k}\right)$$

$$\Rightarrow J_w + B e^{J_w k} - B$$
$$= A\Pi_2 - A\Pi_4 e^{J_w k}.$$

$$B e^{J_w k} + A\Pi_4 e^{J_w k}$$
$$= A\Pi_2 + B - J_w.$$

$$e^{J_w k} = \frac{A\Pi_2 + B - J_w}{A\Pi_4 + B}$$

Therefore,

$$\boxed{K = \frac{1}{J_w} \ln \frac{B + A\Pi_2 - J_w}{B + A\Pi_4}}$$

Note that

$$J_w = A(\Pi_2 - \Pi_3)$$
$$J_w - A\Pi_2 = -A\Pi_3.$$
$$A\Pi_2 - J_w = \underbrace{A\Pi_3 > A\Pi_4}.$$

Therefore, the log term gives you a positive number.

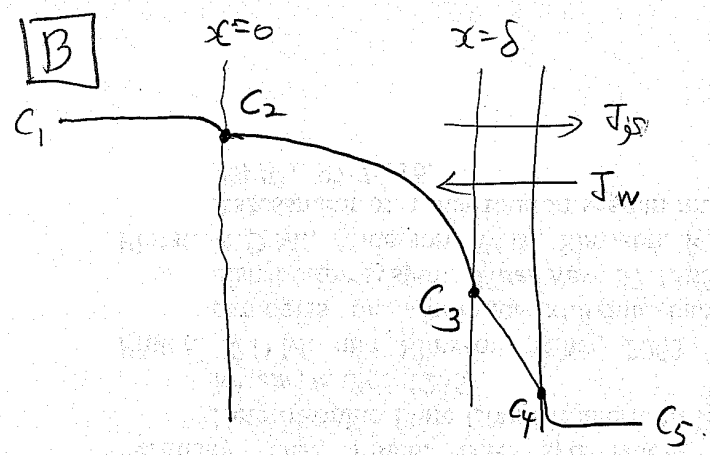
Replacing $\Pi_2 = \Pi_{high}$ & $\Pi_4 = \Pi_{low}$. assuming

$$\boxed{K = \frac{1}{J_w} \ln \frac{B + A\Pi_{high} - J_w}{B + A\Pi_{low}}}$$

$C_1 \simeq C_2 \Rightarrow \Pi_{high}$

$C_5 \simeq C_5 \Rightarrow \Pi_{low}$.

$\boxed{B}$



- Water flux : $J_w = A(\Pi_3 - \Pi_4)$ — (8)
- Solute flux : $J_s = B(C_3 - C_4)$ — (9)

Since C_3 is unknown, it is needed to be represented in terms of C_2 and C_4

- Mass balance equation.

$$+J_s = -D\frac{\varepsilon}{\tau}\frac{dc}{dx} - J_w C \quad \text{— (10)}$$

because it has the same mathematical form as Eq. (3), the general solution is

$$C = -\frac{J_s}{J_w} + C_0'' e^{-\frac{J_w \tau x}{D\varepsilon}}$$

- Boundary conditions are

$$\left.\begin{aligned} C(x=0) &= C_2 \\ C(x=\delta) &= C_3 \end{aligned}\right\} \quad (11)$$

- B.C 1

$$C_2 = -\frac{J_s}{J_w} + C_0'' \cdot 1$$

$$C_0'' = C_2 + J_s/J_w$$

- B.C 2

$$C_3 = -\frac{J_s}{J_w} + \left(C_2 + \frac{J_s}{J_w}\right)e^{-J_w K}$$

where $K = \frac{\tau\delta}{D\varepsilon}$

$$\boxed{C_3 = \left(\frac{J_s}{J_w}\right)\left(e^{-J_w K} - 1\right) + C_2 e^{-J_w K}} \quad (12)$$

↓ continue

$$J_w C_3 = B(C_3 - C_4)(e^{-J_w K} - 1) + J_w C_2 e^{-J_w K}$$

$$= B C_3 [e^{-J_w K} - 1] - B C_4 [e^{-J_w K} - 1] + J_w C_2 e^{-J_w K}$$

$$J_w C_3 - B C_3 [e^{-J_w K} - 1] = J_w C_2 e^{-J_w K} - B C_4 [e^{-J_w K} - 1]$$

$$C_3 [J_w + B[1 - e^{-J_w K}]] = J_w C_2 e^{-J_w K} + B C_4 [1 - e^{-J_w K}]$$

$$\boxed{C_3 = \frac{J_w C_2 e^{-J_w K} + B C_4 (1 - e^{-J_w K})}{J_w + B(1 - e^{-J_w K})}} \quad (13)$$

Now back to water flux:

$$J_w = A(\Pi_3 - \Pi_4)$$

$$\frac{J_w}{A} = (\Pi_2 - \Pi_4)\frac{\Pi_3 - \Pi_4}{\Pi_2 - \Pi_4}$$

$$= \Pi_2\left(1 - \frac{\Pi_4}{\Pi_2}\right)\frac{C_3 - C_4}{C_2 - C_4}$$

$$= \Pi_2\left(1 - \frac{\Pi_4}{\Pi_2}\right)\frac{C_3 - C_4}{C_2(1 - C_4/C_2)}$$

$$= \Pi_2 \frac{C_3 - C_4}{C_2}$$

$$= \left(\frac{\Pi_2}{C_2}\right)(C_3 - C_4)$$

$$\frac{J_w C_2}{A\Pi_2} = C_3 - C_4$$

$$= \frac{J_w C_2 e^{-J_w k} + BC_4(1 - e^{-J_w k}) - C_4 J_w - BC_4(1 - e^{-J_w k})}{J_w + B(1 - e^{-J_w k})}$$

$$= \frac{J_w C_2 e^{-J_w k} - J_w C_4}{J_w + B(1 - e^{-J_w k})}$$

$$\Rightarrow \quad J_w = \frac{A\Pi_2}{C_2}\,\frac{C_2 e^{-J_w k} - C_4}{1 + BJ_w^{-1}(1 - e^{-J_w k})}$$

$$= A\Pi_2\,\frac{1 - \frac{C_4}{C_2}e^{J_w k}}{e^{J_w k} + BJ_w^{-1}(e^{J_w k} - 1)}$$

Finally,

$$\boxed{J_w = A\Pi_2\,\frac{1 - (C_4/C_2)e^{J_w k}}{e^{J_w k} + BJ_w^{-1}(e^{J_w k} - 1)}} \quad \text{— (13)}$$

From eq. (13)

$$\frac{J_w}{A\pi_2}\left[e^{J_w k} + BJ_w^{-1}(e^{J_w k}-1)\right] = 1 - \frac{c_4}{c_2}e^{J_w k}$$

$$\frac{c_4}{c_2} \simeq \frac{\pi_4}{\pi_2}$$

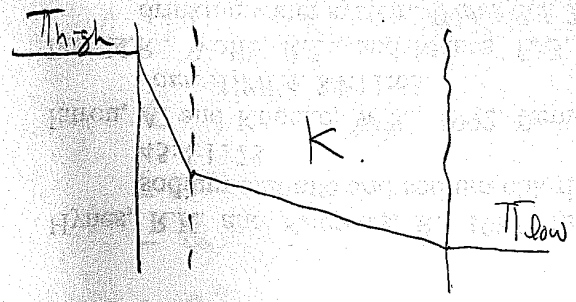
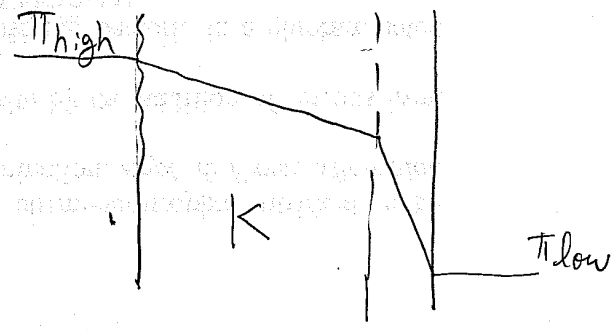
$$\frac{J_w}{A\pi_2}e^{J_w k} + \frac{B}{A\pi_2}e^{J_w k} - \frac{B}{A\pi_2} = 1 - \frac{c_4}{c_2}e^{J_w k}$$

$$e^{J_w k}\left[\frac{J_w}{A\pi_2} + \frac{B}{A\pi_2} + \frac{A\pi_4}{A\pi_2}\right] = 1 + \frac{B}{A\pi_2} = \frac{A\pi_2 + B}{A\pi_2}$$

$$e^{J_w k} = \frac{B + A\pi_2}{B + A\pi_4 + J_w}$$

Therefore

$$\boxed{K = \frac{1}{J_w}\ln\frac{B + A\pi_{high}}{B + J_w + A\pi_{low}}}$$

Summary

• draw – membrane – support – feed (DMSF)	• draw – support – membrane – feed (DSMF)
$K = \frac{1}{J_w}\ln\frac{B + A\pi_{high} - J_w}{B + A\pi_{low}}$	$K = \frac{1}{J_w}\ln\frac{B + A\pi_{high}}{B + J_w + A\pi_{low}}$
where $\pi_{high} \cong \pi_{draw}$ & $\pi_{low} \cong \pi_{feed}$