Effect of porous support fabric on osmosis through a Loeb–Sourirajan type asymmetric membrane

Sidney Loeb*, Leonid Titelman, Emmanuel Korngold, Joseph Freiman

Institutes for Applied Research, Ben-Gurion University of the Negev, Beer Sheva, Israel

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Abstract

Commercially available asymmetric membranes of the Loeb–Sourirajan (L–S) type comprise a support fabric, bonded to the porous substructure. The influence of this fabric on osmotic permeation flux was examined, mostly with a Toray CA-3000 membrane from which, with care, it was possible to remove the support fabric. In osmosis experiments with 12% MgCl₂ solution on one side (either side) and 6% solution on the other, the permeation flux (J₁) was of the order of 0.01 and 0.06 m³/m²·d with and without fabric, respectively. These results could be generalized by considering the resistivity to solute diffusion in the non-skin part of the membrane. This resistivity term averaged 104 and 17 d/m for membranes with and without fabric, respectively, and in further tests without fabric, it was between 15 and 25 d/m over a wide range of MgCl₂ concentrations.

Four other L–S membranes, all with support fabric, were tested in osmosis experiments. Their resistivity values were similar to or higher than those of the Toray membrane with fabric, but, with one of the four, the results were affected by switching the location of the high and low concentration solutions. It was concluded that existing commercially available L–S membranes are not appropriate for large-scale osmosis applications because their support fabric decreases permeation flux excessively.

Keywords: Asymmetric membranes; Concentration polarization; Loeb–Sourirajan membranes; Membrane preparation and structure; Osmosis

1. Introduction

The practical large-scale use of pressure-retarded osmosis (PRO) has been considered by previous investigators [1,2] who concluded that among the commercially available flat sheet membranes, the Loeb–Sourirajan (L–S) type holds the most promise for osmosis because it offers the highest permeation flux for a given osmotic driving force.

*Corresponding author. PO Box 41, Omer, 84965 Israel. Tel.: 972-76460475; fax: 972-76467763; e-mail: sidloeb@bgumail.bgu.ac.il.

These investigators also found that the porous substructure of such membranes is the principal hindrance to osmotic flow. However, their work was always confined to membranes without the woven or non-woven porous support fabric which is universally applied today to all commercially available L–S membranes by bonding to the substructure side.

Since the porous substructure is the principal resistance to osmotic permeation through a membrane without a support fabric, one must ask whether the addition of a fabric on the porous substructure side would act as an excessive additional resistance. The
Fig. 1. Transport conditions in (A) reverse osmosis and (B) osmosis, both through an asymmetric membrane.
2. Comparison of transport in reverse osmosis and osmosis, both through an asymmetric membrane without a support fabric

Fig. 1A shows the pressure and solute gradients across an L–S membrane in reverse osmosis (RO). The water permeation flux, $J_1$, is taken from the high concentration solution, flows to the left from the skin into the porous substructure, and passes through it with no change in the solute concentration, $C_2$, and little change in pressure. Hence, the porous substructure thickness plays only a minor role in determining the magnitude of $J_1$.

In osmosis, the situation has been described by Lee et al. [2] as internal polarization. As shown in Fig. 1B, the water permeation flux, $J_1$, is taken from the low concentration solution, flows to the right in the porous substructure, carrying the solute with it. At any given point in the porous substructure, where the solute concentration is $C_2_{ps}$, the solute flux, $J_2$, tends to flow to the right, while coupled to $J_1$, at a rate $J_2 = J_1 C_2_{ps}$. The solute flux is largely stopped by the skin at the transition interface, causing the solute concentration there, $C_2$, to rise. However, this rise causes back diffusion of the solute to the left at a rate $J_2 = -(D_{2ps}) (dC_2/dx)$ where $D_{2ps}$ is the diffusivity of the solute in the porous substructure.

The solute concentration, $C_2$, continues to rise until a steady state is reached at $\pi_{tr}$ and $C_{2tr}$. Since $\pi_{tr}$ is greater than $\pi_{Low}$, the skin transport equations in Fig. 1B show that $J_1$ (actual) is less than $J_1$ (ideal). $J_1$ (actual) can only approach closely to $J_1$ (ideal) if the solute back diffusion rate is high or the diffusion thickness, $t$, is low. This is equivalent to saying that the solute resistivity $K (= t/D_{2ps})$ in the substructure is low. Thus, in osmosis, the porous substructure clearly has a large effect in determining the water permeation flux, $J_1$.

The exact equation, developed by Lee et al. [2], is given in Appendix A for the case in which the low concentration solution is on the porous substructure side. If the not-always-precise assumption, $C_{2A}/C_{2B} = \pi_{A}/\pi_{B}$, is made and the experimentation is restricted to the condition of appreciable osmotic pressure on both sides of the membrane, the Lee equation can be reduced to Eq. (A3) of Appendix A:

$$K = (1/J)\ln(\pi_{Hi}/\pi_{Low})$$

It appears from further analysis that this equation would apply whether the low concentration solution is on the skin side or the porous substructure side of the membrane. It should also be noted that the osmotic driving force is essentially the logarithm of the ratio of high to low osmotic pressure. In RO, the osmotic driving force is the difference between high and low osmotic pressures.

The above discussion and the analysis in Appendix A are for the asymmetric membrane without a support fabric, in which case the solute resistivity term, $K$, refers to solute diffusion in the porous substructure. When the support fabric is included, there are, as shown in Fig. 2, three distinct regions for solute diffusion specifically: most of the substructure, most of the support fabric, and an overlap region combining the first two.

The methods of testing and data analysis used here are also appropriate for an overall solute resistivity

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Footnote 1: Fig. 1B also gives nomenclature used throughout this paper with the exception of $K_{lab}$, to be defined in this section.

Fig. 2. Edgewise dimensions of the Toray CA-3000 membrane.
term to be used with the membranes having a support fabric. This resistivity term will be called $K_{fub}$.

3. Description of Toray CA-3000 membrane

The Toray CA 3000 membrane is an RO cellulose acetate asymmetric membrane supported by a woven fabric. It was chosen for extensive tests because the support fabric could, with care, be removed from the rest of the membrane. This enabled comparison of performance with and without the fabric. In an edge-wise view, the membrane and support material have the dimensions shown in Fig. 2.

The characteristic performance of the membrane in RO, according to Toray, is: fractional NaCl rejection, $R$, 0.98 (98%); water flux, $J_1$, 0.90 m$^3$/m$^2$ d. Data were obtained at a feed concentration of 1500 mg/l NaCl, and operating pressure of 30 kg/cm$^2$ (or 29.03 atm or 29.41 bar or 426.8 psi).

NaCl at 1500 mg/l has 1.2 atm osmotic pressure. Then, in RO:

$$A = J_1/(\Delta P - \Delta \pi) = 0.9/(29.03 - 1.2) = 0.0328 \text{ m}^3/\text{m}^2 \text{ d atm.}$$

$$B = J_2/(C_{2hi} - C_{2low}).$$

$A$ is the water permeation constant, in m/d. If $C_2$ has the units kg of solute per m$^3$ then $J_2$ is kg of solute per m$^2$ per day. It can be shown [2] that:

$$B = \frac{(1 - R)(A)(\Delta P - \Delta \pi)}{R} = \frac{(1 - R)J_1}{R} = \frac{(1 - 0.98)(0.90)}{0.98} = 0.0184 \text{ m/d}$$

4. Test apparatus and procedure

The osmosis test apparatus is shown conceptually in Fig. 3. As a batch device, conditions were changing constantly during the course of a test. Specifically, the dilute solution was concentrating and the other was diluting. Hence, the osmotic driving force, $\ln(\pi_{Hi}/\pi_{Low})$, was continually diminishing, producing a corresponding decrease in the water permeation flux, $J_1$, in accordance with Eq. (A3) in Appendix A:

$$J_1 \approx \ln(\pi_{Hi}/\pi_{Low})$$

The tests were short enough so that the arithmetic average values were acceptable, i.e. $\ln(\pi_{Hi,ave}/\pi_{Low,ave})$. $J_1$ was determined of course from the change in burette levels. The tests reported herein were all made with appreciable magnesium chloride concentrations on both sides of the membranes. The con-

![Osmosis test cell](image)

Conditions tested:
1) Low concentration solution on porous substructure side or on skin side.
2) Solution pairs 6-12% (as shown) or 6-18, 12-18, 12-24% MgCl$_2$.
3) Membrane with or without fabric.

Fig. 3. Osmosis test cell.
centrations and therefore osmotic pressures were readily known from density measurements of solutions both before and after the tests.

5. Purposes of tests

The primary purpose of the tests was to determine the effect of the support fabric on osmotic water permeation flux. The value of the solute resistivity term, $K$ (or $K_{fab}$), was also determined for each test. Since $K$ is a property dependent primarily on the nature of the solute and the porous medium through which it diffuses, $K$ should be relatively constant for a given solute whatever the value of $J_1$, $\pi_{Hi}$, and $\pi_{Low}$ in a given test, i.e. $K$ should be an appropriate parameter for evaluating osmosis membranes. Hence, another purpose of these tests was to determine the constancy of $K$ with varying values of $J_1$, $\pi_{Hi}$, and $\pi_{Low}$.

Another tentative conclusion reached in the development of Eq. (A3) was that it should be valid whether the low concentration solution is on the skin side or the opposite side of the membrane, with or without fabric. This conclusion could also be examined by the test procedures, i.e. by simply switching the solutions from one side to the other.

6. Test results with Toray cellulose acetate membrane, CA-3000

The following conclusions can be gleaned from the tests reported in Table 1:

First, the detrimental effect of the support fabric on the water permeation flux is very clear. In the tests with the 12/6 concentration pair the average permeation flux was ~1/6 as high with the fabric as without it (0.01 vs. 0.06 m$^3$/m$^2$ d). Approximately, this same value is seen in the 17/104 ratio for $K/K_{fab}$.

Second, in all the tests made without the support fabric, the term $K$ was relatively constant at values between 15 and 25 d/m, regardless of appreciable variation in $J_1$, $\pi_{Hi}$, and $\pi_{Low}$, and their ratio. Thus, Eq. (A3) is supported and therefore the expectation

<table>
<thead>
<tr>
<th>Test dates (November 1995)</th>
<th>Approximate initial MgCl$_2$ solutions on each side of the membrane</th>
<th>Water permeation flux $J_1$ (m$^3$/m$^2$ d)</th>
<th>Resistivity to MgCl$_2$ diffusion (d/m)</th>
<th>Solution in contact with skin side, Hi or Low concentration</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12/6</td>
<td>140/50</td>
<td>0.012</td>
<td>Hi</td>
</tr>
<tr>
<td>22</td>
<td>12/6</td>
<td>0.0097</td>
<td>0.012</td>
<td>Hi</td>
</tr>
<tr>
<td>20</td>
<td>12/6</td>
<td>0.0097</td>
<td>0.012</td>
<td>Hi</td>
</tr>
<tr>
<td>8</td>
<td>12/6</td>
<td>140/50</td>
<td>0.060</td>
<td>18 Hi</td>
</tr>
<tr>
<td>5</td>
<td>12/6</td>
<td>0.056</td>
<td>15 Low</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>12/6</td>
<td>0.051</td>
<td>18 Hi</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>12/6</td>
<td>0.071</td>
<td>15 Hi</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>12/6</td>
<td>0.064</td>
<td>18 Hi</td>
<td></td>
</tr>
<tr>
<td>Ave. 17</td>
<td></td>
<td>Av. 17</td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>12/6</td>
<td>290/50</td>
<td>0.094</td>
<td>17 Hi</td>
</tr>
<tr>
<td>13</td>
<td>12/6</td>
<td>290/140</td>
<td>0.020</td>
<td>25 Hi</td>
</tr>
<tr>
<td>23</td>
<td>12/6</td>
<td>0.031</td>
<td>23 Hi</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>24/12</td>
<td>540/140</td>
<td>0.059</td>
<td>20 Hi</td>
</tr>
</tbody>
</table>

*Tests on 8, 9, 12, 13, 15 and 16 November were all made with the same membrane.
that the term $K$ is dominant and general in determining the osmotic permeation flux.

Third, the tests on 22nd and 5th November were done with the low concentration solution on the skin side. The approximate equality of $K$ and $K_{fab}$ values with those of the other tests in their respective groups supported the belief that the side of the membrane in contact with the low concentration solution is not important (but see the following section).

7. Tests with other cellulose acetate membranes

Osmosis tests with 6 and 12% MgCl$_2$ solutions were made on several other commercially available cellulose acetate membranes. All of these were backed with woven or non-woven fabric. No effort was made to remove the fabric.

The results given in Table 2 show that the $K_{fab}$ values are in the range of those with the fabric-backed Toray membrane or higher. As with the Toray membrane, the results were independent of the location of the 6 and 12% MgCl$_2$ solutions for the Fluid Systems', Trisep's, and Desalination Systems' membranes. However, the Hydranautics' membrane had roughly twice the $K_{fab}$ value when the low concentration solution was against the skin as when the solutions were reversed. This anomalous behavior was not explored further.

Table 2
Solute resistivity ($K_{fab}$) values obtained with fabric-backed cellulose acetate membranes other than Toray CA-3000

<table>
<thead>
<tr>
<th>Manufacturer</th>
<th>Designation</th>
<th>$K_{fab}$ (d/m)</th>
<th>Commentary</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fluid Systems</td>
<td>LP</td>
<td>82</td>
<td>Note 1</td>
</tr>
<tr>
<td></td>
<td>CA/CTA</td>
<td>96</td>
<td>Note 2</td>
</tr>
<tr>
<td>Trisep</td>
<td>None</td>
<td>140</td>
<td>Note 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>140</td>
<td>Note 2</td>
</tr>
<tr>
<td>Desalination Systems</td>
<td>CD-Yuma</td>
<td>380</td>
<td>Note 1</td>
</tr>
<tr>
<td></td>
<td>CA/CTA</td>
<td>380</td>
<td>Note 2</td>
</tr>
<tr>
<td>Hydranautics</td>
<td>CAB-1</td>
<td>246, 204</td>
<td>Note 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>93, 136</td>
<td>Note 2</td>
</tr>
</tbody>
</table>

Note 1: Low concentration solution against skin.  
Note 2: Low concentration solution against fabric.  
All tests done with 6 and 12% MgCl$_2$.

8. Discussion and conclusions

Clearly, the porous support fabric on L-S membranes decreases osmotic permeation excessively. However, without the fabric the membrane is too weak. A compromise would appear to be the use of a much more open fabric. However, the unanimous opinion of the manufacturers is that a more open fabric would cause ‘bleeding’ through it during fabrication, and would decrease the reliability of the completed membrane.

As touched upon by Lee et al. [2], an L-S capillary tube might be appropriate since it would not require further support. The internal diameter should be at least 100 µm to minimize axial pressure drop within the capillary tube. The wall thickness should be great enough to maintain capillary rigidity but otherwise, thin enough to achieve adequate permeation flux since $K$ is proportional to porous substructure thickness. A module filled with capillary tubing would be satisfactory for osmosis with non-precipitating solutions. However, in one possible application, the concentration of Dead Sea brine, extensive precipitation of NaCl would occur as the brine is dewatered because it is saturated to start. Therefore, the module would have to be designed to pass a slurry of concentrated Dead Sea brine and NaCl without clogging, which is not an easy task.

Acknowledgements

We wish to thank the Toray, Fluid Systems, Trisep, and Hydranautics companies for the opportunity to test their membranes.

Appendix A

Key equations without support fabric

A.1 Low concentration solution on porous substructure side of membrane

Eq. (10) in [2] is:

$$
\frac{J_1}{A \pi_{HI}} = \frac{1 - (C_{2L\text{Low}}/C_{2H\text{Hi}})\exp(J_1 K)}{1 + B/J_1[\exp(J_1 K) - 1]}
$$

(A1)

In this equation, $A$ and $B$ are known from the RO experiments; $\pi_{HI}$, $C_{2L\text{Low}}$, and $C_{2H\text{Hi}}$ are given conditions.
in the osmosis experiment; and \( J_1 \) is measured during the experiment. Therefore, the equation can be used to determine \( K \), defined as \( t/D_2ps \).

By using the rough but simplifying assumption that for a given solute in a solution, \( \pi_A/\pi_B = C_{2A}/C_{2B} \), Eq. (A1) can be simplified to:

\[
K = \left( \frac{1}{J_1} \right) \left( \frac{B + A\pi_{Hi} - J_1}{B + A\pi_{Low}} \right) \tag{A2}
\]

In our tests, \( \pi_{Low} \) was always appreciably higher than zero. Also, \( A\pi_{Low} \) and, therefore, \( A\pi_{Hi} \) were much higher than \( B \) or \( J \), so that we can write:

\[
K = \frac{1}{J_1} \left( \frac{\pi_{Hi}}{\pi_{Low}} \right) \tag{A3}
\]

### A.2 Low concentration solution on skin side of membrane

By an analysis similar to that in Section A.2 above, we arrive at:

\[
K = \left( \frac{1}{J_1} \right) \left( \frac{B + A\pi_{Hi}}{B + J_1 + A\pi_{Low}} \right) \tag{A4}
\]

Using the same simplifying assumptions that led to Eq. (A3), we arrive at the same equation. Therefore, the same value of \( K \) should be obtained whichever side of the membrane is in contact with the low concentration solution. Note that Eq. (A3) can be written to emphasize the inverse effect of substructure thickness \( t \) on \( J_1 \):

\[
J_1 = \frac{1}{K} \left( \frac{\pi_{Hi}}{\pi_{Low}} \right) = \frac{D_2ps}{t} \left( \frac{\pi_{Hi}}{\pi_{Low}} \right)
\]

### References


Internal Concentration Polarization of Forward Osmosis.

A. Lee et al.

\[ \text{1st Diagram:} \]

- \( J_w = \text{water flux} \) in magnitude
- \( J_s = \text{solute flux} \) in magnitude

A. Lee et al. : \( J_w = ? \)

- \( J_w = A \Delta \Pi_{eff} = A (\Pi_a - \Pi_b) \quad \text{(1)} : \text{water flux} \)
- \( J_s = B (C_s - C_b) \quad \text{(2)} : \text{solute flux} \)
- In support layer
  \( J_s = -D \frac{\delta}{\delta x} \frac{dc}{dx} - J_w C \quad \text{(3)} : \text{Convective diffusion eq.} \)

Solution:

\[ D \frac{\delta}{\delta x} \frac{dc}{dx} = -J_w C - J_s \]

\[ \frac{dc}{dx} = - \frac{\delta J_w}{\delta x} (C + \frac{J_s}{J_w}) \]

\[ \Rightarrow \frac{dc}{dx} = - \frac{\delta J_s}{\delta x} C' \]

f. \( C' = C_0 - \frac{J_s}{J_w} \frac{\delta C}{\delta x} \)

\[ C = - \frac{J_s}{J_w} C_0 + C_0'' e^{- \frac{J_s}{J_w} \frac{\delta x}{\delta x}} \]

B.C. \( \text{at} \ x = 0 \)

- \( - \frac{J_s}{J_w} + C_0'' C = C_3 \)
- \( C_0'' = C_3 + \frac{J_s}{J_w} \)

\[ \Rightarrow C(x = 0) = - \frac{J_s}{J_w} + (\frac{J_s}{J_w} + C_3) e^\frac{-J_s}{J_w} \frac{\delta x}{\delta x} \]

B.C. \( \text{at} \ x = \delta \)

- \( \frac{J_s}{J_w} + (\frac{J_s}{J_w} + C_3) e^\frac{-J_s}{J_w} \frac{\delta x}{\delta x} \)

\[ \Rightarrow C(x = \delta) = C_4 \]

\[ C_4 = - \frac{J_s}{J_w} + (\frac{J_s}{J_w} + C_3) e^\frac{-J_s}{J_w} \]

\[ \text{where} \quad K = \frac{\delta J_s}{\delta x} \frac{\delta x}{\delta x} \]

\[ C_4 = (\frac{J_s}{J_w}) (\delta e^\frac{-J_s}{J_w} \delta x) + \frac{\delta e^\frac{-J_s}{J_w} \delta x}{K} \]

- \( \text{(4)} \)
\[ J_w = A \left( \pi_2 - \pi_3 \right) \]

Assume \( \frac{\pi_2 - \pi_3}{\pi_2 - \pi_f} = \frac{c_2 - c_3}{c_2 - c_f} \)

\[ = A \cdot \frac{c_2 - c_3}{c_2 - c_f} \left( \pi_2 - \pi_f \right) \]

\[ = A \cdot \frac{\pi_3 (1 - \frac{\pi_f}{\pi_3})}{(1 - c_f/c_3)} \left( 1 - \frac{c_3}{c_2} \right) \]

\[ = A \pi_2 \left( 1 - \frac{c_3}{c_2} \right) \]

From Eq. (4),

\[ C_4 = \frac{J_w}{J_3} (e^{-J_w k} - 1) + C_a e^{-J_w k} \]

Then,

\[ J_w = A \pi_2 \left( 1 - \frac{c_3}{c_2} \right) \]

\[ = A \pi_2 \frac{J_w (1 - \frac{c_4}{c_2} e^{J_w k})}{J_w + B (e^{J_w k} - 1)} \]
\[ J_w + B \left[ e^{J_w k} - 1 \right] = A \left( \Pi_2 - \Pi_4 e^{J_w k} \right) \]

\[ \Rightarrow J_w + B e^{J_w k} - B = A \Pi_2 - A \Pi_4 e^{J_w k} \]

\[ B e^{J_w k} + A \Pi_4 e^{J_w k} = A \Pi_2 + B - J_w \]

\[ e^{J_w k} = \frac{A \Pi_2 + B - J_w}{A \Pi_4 + B} \]

Therefore,

\[ K = \frac{1}{J_w} \ln \left( \frac{B + A \Pi_2 - J_w}{B + A \Pi_4} \right) \]

Note that

\[ J_w = A (\Pi_2 - \Pi_3) \]

\[ J_w - A \Pi_2 = -A \Pi_3. \]

\[ A \Pi_2 - J_w = A \Pi_3 \Rightarrow A \Pi_4. \]

Therefore, the log term gives you a positive number.

Replacing \( \Pi_2 = T_{\text{high}} \) & \( \Pi_4 = T_{\text{low}} \), assuming

\[ c_1 \approx c_2 \Rightarrow T_{\text{high}} \]

\[ c_5 \approx c_5 \Rightarrow T_{\text{low}}. \]
\[ \begin{align*} 
\text{B.C.1:} & \quad C_2 = -\frac{J_s}{J_w} + C_0'' \cdot 1 \\
& \quad C_0'' = C_s + J_s/J_w \\
\text{B.C.2:} & \quad C_3 = -\frac{J_s}{J_w} + (C_s + \frac{J_s}{J_w}) e^{-J_w K} \\
& \quad \text{where } K = \frac{a^2 S}{D E}. \\
\end{align*} \]

\[ C_3 = \left( \frac{J_s}{J_w} \right) (e^{-J_w K} - 1) + C_2 e^{-J_w K} \]

\[ \text{C.12) \text{Continue}} \]

\[ \begin{align*} 
J_w C_3 & = B C_3 (e^{-J_w K} - 1) + J_w C_2 e^{-J_w K} \\
& = \beta C_3 \left[ e^{-J_w K} - 1 \right] \\
& \quad - \beta C_2 \left[ e^{-J_w K} - 1 \right] \\
& \quad + J_w C_2 e^{-J_w K}. \\
\end{align*} \]

\[ \begin{align*} 
J_w C_3 - B C_3 \left[ e^{-J_w K} - 1 \right] \\
& = J_w C_2 e^{-J_w K} - B C_2 \left[ e^{-J_w K} - 1 \right] \\
& = J_w C_2 e^{-J_w K} + B C_4 \left[ 1 - e^{-J_w K} \right] \\
& = J_w C_2 e^{-J_w K} + B C_4 \left[ 1 - e^{-J_w K} \right]. \\
\end{align*} \]

\[ C_3 = \frac{J_w C_2 e^{-J_w K} + B C_4 \left[ 1 - e^{-J_w K} \right]}{J_w + B \left[ 1 - e^{-J_w K} \right]}. \]
Now back to water flux: 

\[ J_w = A \left( \Pi_b - \Pi_4 \right) \]

\[
= \frac{J_w}{A} \left( \Pi_b - \Pi_4 \right) \frac{\Pi_b - \Pi_4}{\Pi_b - \Pi_4}
\]

\[
= \Pi_b \left( 1 - \frac{\Pi_4}{\Pi_b} \right) \frac{C_3 - C_4}{C_2 - C_4}
\]

\[
= \Pi_b \left( 1 - \frac{\Pi_4}{\Pi_b} \right) \frac{C_3 - C_4}{C_2 \left( 1 - \frac{C_4}{C_2} \right)}
\]

\[
= \Pi_b \cdot \frac{C_3 - C_4}{C_2
\]

\[
= \left( \frac{\Pi_b}{C_2} \right) \left( C_3 - C_4 \right)
\]

\[
= \frac{J_w C_2}{A \Pi_b} = C_3 - C_4
\]

\[
= \frac{J_w C_2 e^{-J WK}}{J_w + B \left( 1 - e^{-J WK} \right)}
\]

\[
= \frac{J_w C_2 e^{-J WK}}{J_w + B \left( 1 - e^{-J WK} \right)}
\]

\[
= \frac{J_w C_2 e^{-J WK}}{J_w + B \left( 1 - e^{-J WK} \right)}
\]

\[
= \frac{J_w C_2 e^{-J WK} - J_w C_4}{J_w + B \left( 1 - e^{-J WK} \right)}
\]

\[
= \frac{A \Pi_b}{C_2} \frac{C_3 e^{-J WK} - C_4}{1 + B J W^{-1} \left( 1 - e^{-J WK} \right)}
\]

\[
= \frac{A \Pi_b}{C_2} \frac{1 - \frac{C_4}{C_2} e^{J WK}}{e^{J WK} + B J W^{-1} \left( e^{J WK} - 1 \right)}
\]

Finally,

\[
J_w = A \Pi_b \frac{1 - \left( \frac{C_4}{C_2} \right) e^{J WK}}{e^{J WK} + B J W^{-1} \left( e^{J WK} - 1 \right)}
\]
From eq. 13
\[ \frac{J_w}{A T_{L_2}} \left[ e^{J_w k} + B J_w \left( e^{J_w k} - 1 \right) \right] = 1 - \frac{C_y}{c_2} e^{J_w k} = \frac{\Pi_{T_2}}{\Pi_{T_2}}. \]
\[ \frac{J_w}{A T_{L_2}} e^{J_w k} + \frac{B}{A T_{L_2}} e^{J_w k} - \frac{B}{A T_{L_2}} = 1 - \left( \frac{C_y}{c_2} \right) e^{J_w k}. \]
\[ e^{J_w k} \left[ \frac{J_w}{A T_{L_2}} + \frac{B}{A T_{L_2}} + \frac{A T_{T_1}}{A T_{L_2}} \right] = 1 + \frac{B}{A T_{L_2}} = \frac{A T_{T_2} + B}{A T_{L_2}}. \]
\[ e^{J_w k} = \frac{B + A T_{L_2}}{B + A T_{T_1} + J_w}. \]

Therefore

\[ K = \frac{1}{J_w} \ln \frac{B + A T_{T_{high}}}{B + J_w + A T_{T_{low}}}. \]

Summary

<table>
<thead>
<tr>
<th>Draw - Membrane - Support - Feed (DMSF)</th>
<th>Draw - Support - Membrane - Feed (DSMF)</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ K = \frac{1}{J_w} \ln \frac{B + A T_{T_{high}} - J_w}{B + A T_{T_{low}}} ]</td>
<td>[ K = \frac{1}{J_w} \ln \frac{B + A T_{T_{high}}}{B + J_w + A T_{T_{low}}} ]</td>
</tr>
</tbody>
</table>

where \[ T_{T_{high}} \approx T_{draw} \] & \[ T_{T_{low}} \approx T_{feed} \]